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Exact solutions in nonlinear diffusion

J. H. KNIGHT AND J. R. PHILIP

CSIRO Division of Environmental Mechanics, Canberra, Australia

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SUMMARY

A method of linearizing the one-dimensional nonlinear diffusion equation subject to arbitrary initial conditions and a flux boundary condition at the origin is shown to apply if and only if the diffusivity $D(\theta) = a(b-\theta)^{-2}$. θ is concentration; $a(>0)$ and b are constants. The exact solutions which follow are, in general, implicit, but explicit solutions are found for the instantaneous source and for redistribution in the finite region. The source is necessarily distributed with concentration b, in contrast to the classical point source. Redistribution solutions are explicit for all initial distributions which may be represented through truncated Fourier cosine expansions of 4 terms or less.

1. Introduction

The nonlinear diffusion equation arises in many applications, including physical chemistry, heat and mass transfer engineering, the physics of solids, metallurgy, fluid mechanics, soil mechanics, and the earth sciences. Solutions satisfying any well-posed set of conditions are, in principle, always available through the brute-force use of finite-difference methods on high-speed computers: but, for both intellectual and economic reasons, it is desirable to take the study of nonlinear diffusion as far as we can by the methods of mathematical analysis. Analytical and quasi-analytical methods are, in fact, to hand for treating many problems in nonlinear diffusion (*e.g.* [1–4]); for certain problems, however, including those of instantaneous sources, those of redistribution, and those subject to flux boundary conditions, the established armoury of methods and of exact solutions is very meagre. This paper presents an analytical method of attacking these problems, which, in general, yields exact solutions in implicit form, and gives explicit exact solutions in some instances including the problems of the instantaneous source and of redistribution in the finite region, treated in sections 3–4 and 5, respectively. The work entails the reinterpretation and extension of a method due to Storm [5].

2. Linearization procedure

We are concerned with the one-dimensional nonlinear diffusion equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial \theta}{\partial x}\right). \tag{2.1}$$

Initially we take the diffusivity D as an arbitrary non-negative function of concentration θ; t denotes time and x the space coordinate. The Kirchhoff transformation [6]

$$\Theta = \int^{\theta} D(\theta')d\theta' \tag{2.2}$$

reduces (2.1) to

$$\frac{\partial \Theta}{\partial t} = D\frac{\partial^2 \Theta}{\partial x^2}. \tag{2.3}$$

We obtain a linear second-order term through the substitution

$$X(x,t) = \int_0^x D(x',t)^{-\frac{1}{2}}dx', \tag{2.4}$$

which implies

$$\partial/\partial x = D^{-\frac{1}{2}}\partial/\partial X .$$

We find that

$$D\frac{\partial^2\Theta}{\partial x^2} = \frac{\partial^2\Theta}{\partial X^2} - \tfrac{1}{2}D^{-1}\frac{dD}{d\Theta}\left(\frac{\partial\Theta}{\partial X}\right)^2 \tag{2.5}$$

and that

$$\begin{aligned}\left(\frac{\partial\Theta}{\partial t}\right)_x &= \left(\frac{\partial\Theta}{\partial t}\right)_X + \frac{\partial\Theta}{\partial X}\cdot\frac{\partial X}{\partial t}\\ &= \left(\frac{\partial\Theta}{\partial t}\right)_X - \frac{1}{2}\frac{\partial\Theta}{\partial X}\int_0^x D^{-\frac{3}{2}}\frac{dD}{d\Theta}\cdot\frac{\partial\Theta}{\partial t}(x',t)dx'\\ &= \left(\frac{\partial\Theta}{\partial t}\right)_X - \frac{1}{2}\frac{\partial\Theta}{\partial X}\int_0^x D^{-\frac{1}{2}}\frac{dD}{d\Theta}\frac{\partial^2\Theta}{\partial x^2}(x',t)dx' .\end{aligned} \tag{2.6}$$

The suffixes distinguish between the derivative at fixed x and that at fixed X.

Now the integral of (2.6) can be evaluated in terms of $\partial\Theta/\partial x$ *if and only if*

$$D^{-\frac{1}{2}}dD/d\Theta = \text{constant} = 2m . \tag{2.7}$$

Under this condition the integral reduces to

$$2m[\partial\Theta/\partial x - (\partial\Theta/\partial x)_{x=0}] .$$

We discard at once the trivial case $m=0$, i.e., D=constant. When (2.7) holds, (2.5) and (2.6) become, respectively,

$$D\frac{\partial^2\Theta}{\partial x^2} = \frac{\partial^2\Theta}{\partial X^2} - mD^{-\frac{1}{2}}\left(\frac{\partial\Theta}{\partial X}\right)^2$$

and

$$\left(\frac{\partial\Theta}{\partial t}\right)_x = \left(\frac{\partial\Theta}{\partial t}\right)_X - mD^{-\frac{1}{2}}\left(\frac{\partial\Theta}{\partial X}\right)^2 + m\left(\frac{\partial\Theta}{\partial x}\right)_{x=0}\frac{\partial\Theta}{\partial X} .$$

Putting these in (2.3) yields the *linear* equation

$$\frac{\partial\Theta}{\partial t} = \frac{\partial^2\Theta}{\partial X^2} + mf\frac{\partial\Theta}{\partial X} . \tag{2.8}$$

We note that $-(\partial\Theta/\partial x)_{x=0}=f$, the flux density at $x=0$, and that this may be either constant or a function of t. Evidently the problem of solving nonlinear equation (2.1) subject to arbitrary initial conditions plus a flux boundary condition at $x=0$ is reducible to that of solving linear equation (2.8). It remains to determine the permissible $D(\theta)$ functions. Since, from (2.2), $d\Theta/d\theta=D$, (2.7) reduces to

$$D^{-\frac{3}{2}}dD/d\theta = 2m .$$

Integration yields

$$D = a(b-\theta)^{-2} , \tag{2.9}$$

where $a=m^{-2}$ and b is a second constant. Equation (2.8) therefore becomes

$$\frac{\partial\Theta}{\partial t} = \frac{\partial^2\Theta}{\partial X^2} + a^{-\frac{1}{2}}f\,\frac{\partial\Theta}{\partial X} ,$$

and it follows from (2.9) that we may write (2.2) and (2.4) as

$$\Theta = a(b-\theta)^{-1} ,$$

$$X(x,t) = a^{-\frac{1}{2}}\int_0^x [b-\theta(x',t)]\,dx' .$$

Storm [5] had found that the nonlinear heat-conduction equation in the semi-infinite medium subject to an initial uniform temperature condition and a flux boundary condition at $x=0$ may be linearized when the temperature-dependent thermal parameters obey a certain relationship. Knight [7] recognized that elementary transformations reduce all such nonlinear heat-conduction equations to the form (2.1) with D satisfying (2.9), and that linearization carried over to problems with arbitrary initial distributions of θ and time-dependent f. The foregoing development of the linearization procedure is essentially that of [7], but it is somewhat simpler. Knight [7] assumed (2.9), whereas we show here that the linearization applies *only* to $D(\theta)$ functions satisfying (2.9) and to *no others.*

Knight [7] obtained the solution for arbitrary initial distribution and constant f. The general solution is in implicit form and involves complicated integrals. The remainder of this paper is devoted to a study of two interesting special problems with $f=0$, which lead to explicit solutions in terms of elementary functions.

In the succeeding sections we assume that $\theta \leqq b$, the equality applying only to the instantaneous source at $t=0$. The results hold also for $\theta \geqq b$ (as may be checked by replacing θ everywhere by $2b-\theta$), though some minor physical reinterpretation is needed: for example, the "source" of section 3 becomes a "sink". We note further that, although we shall normally think of b as positive (especially with $\theta \leqq b$), this is inessential to the analysis.

3. Nonlinear diffusion from an instantaneous source

The only exact nonlinear instantaneous source solutions available to date have been the similarity solutions which hold when $D \propto \theta^n (n>0)$ and the initial concentration $\theta_0=0$ everywhere [8, 9, 10]. We here develop the explicit exact solution for D satisfying (2.9) and the initial concentration away from the source θ_0 ($<b$) uniform but arbitrary. The advantages of the present solution are, firstly, that (2.9) is a two-parameter form well-adapted to fitting empirical diffusivity data [7, 11]; and, secondly, that it is free of the mathematically special and physically restrictive requirement of previous solutions that $D(\theta_0)=0$.

We study the instantaneous source of magnitude $2Q$ in the infinite region $-\infty \leqq x \leqq +\infty$. Our analysis applies also to the corresponding "half-problem" of the instantaneous source Q in the semi-infinite region $x \geqq 0$. We seek the solution of (2.1), (2.9) describing the diffusion which follows the release at instant $t=0$ of quantity $2Q$ of diffusate in a small region⋆ centered on $x=0$.

Equation (2.1) is thus subject to the condition

$$\begin{aligned} t = 0\,,\ |x| \leqq x_*\,,\ \ \theta &= \theta_0 + Q x_*^{-1}\,, \\ |x| > x_*\,,\ \ \theta &= \theta_0\,. \end{aligned} \tag{3.1a}$$

The required value of x_* emerges in the course of the analysis. Because of symmetry, the required solution satisfies the condition

$$t \geqq 0\,,\ \ x = 0\,,\ \ \partial\theta/\partial x = 0\,. \tag{3.1b}$$

The substitutions

$$\vartheta = \frac{b-\theta}{b-\theta_0}\,,\quad \xi = \frac{b-\theta_0}{Q}\,x\,,\quad \xi_* = \frac{b-\theta_0}{Q}\,x_*\,,\quad \tau = \frac{a}{Q^2}\,t\,, \tag{3.2}$$

reduce (2.1), (2.9), (3.1a), (3.1b) to

$$\frac{\partial\vartheta}{\partial\tau} = \frac{\partial}{\partial\xi}\left(\vartheta^{-2}\frac{\partial\vartheta}{\partial\xi}\right); \tag{3.3}$$

⋆ The classical abstraction of an instantaneous *point* source with infinite initial concentration cannot be realized for this mode of nonlinear diffusion. The instantaneous source is required to be distributed over a finite nonzero region, with finite initial concentration (subsection 4.1).

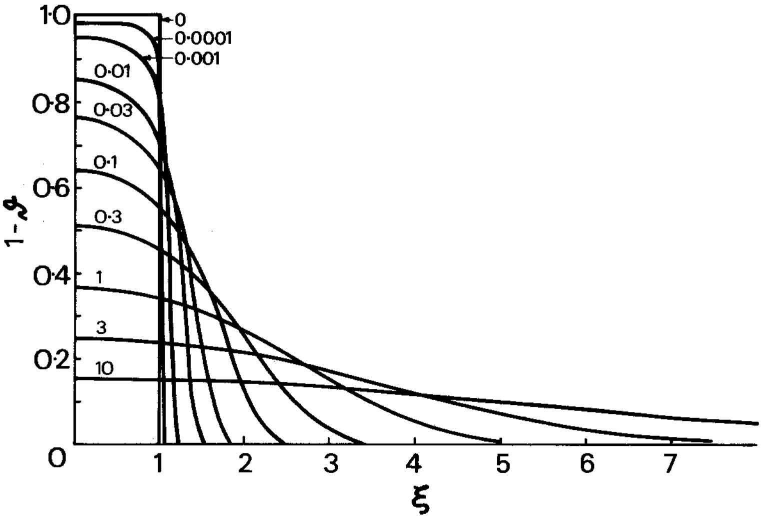

Figure 1. Nonlinear diffusion from an instantaneous source. The time course of concentration in the dimensionless form $1-\vartheta(\xi, \tau)$. Note that $1-\vartheta=(\theta-\theta_0)/(b-\theta_0)$. ξ is the reduced space coordinate. Numerals on the curves denote values of the reduced time τ.

$$\begin{aligned} \tau = 0\,, \quad |\xi| \leqq \xi_*\,, \quad & \vartheta = 1-\xi_*^{-1}\,, \\ |\xi| > \xi_*\,, \quad & \vartheta = 1\,; \end{aligned} \tag{3.4a}$$

$$\tau \geqq 0\,, \quad \xi = 0\,, \quad \partial\vartheta/\partial\xi = 0\,. \tag{3.4b}$$

Following section 2, we linearize through the transformations

$$\Theta = \vartheta^{-1}\,, \tag{3.5}$$

$$\Xi(\xi, \tau) = \int_0^\xi \Theta^{-1}(\xi', \tau)\, d\xi'\,. \tag{3.6a}$$

The inverse of (3.6a) is

$$\xi(\Xi, \tau) = \int_0^\Xi \Theta(\Xi', \tau)\, d\Xi'\,. \tag{3.6b}$$

In view of (3.4b) the required equation is

$$\frac{\partial\Theta}{\partial\tau} = \frac{\partial^2\Theta}{\partial\Xi^2}\,. \tag{3.7}$$

The solution of (3.7) describing an instantaneous point source of magnitude 2 in the infinite region with initial unit concentration everywhere except at the origin is [12, p. 50]

$$\Theta = 1+(\pi\tau)^{-\frac{1}{2}} \exp(-\Xi^2/4\tau)\,. \tag{3.8}$$

This solution is for initial condition

$$\tau = 0\,, \quad -\infty \leqq \Xi \leqq +\infty\,, \quad \Theta = 1+2\delta(\Xi)\,, \tag{3.9}$$

with δ the Dirac delta function. Transforming (3.9) through (3.6b) back to a condition on $\vartheta(\xi, 0)$, we obtain

$$\begin{aligned} \tau = 0\,, \quad |\xi| \leqq 1\,, \quad & \vartheta = 0\,, \\ |\xi| > 1\,, \quad & \vartheta = 1\,. \end{aligned} \tag{3.10}$$

This is exactly (3.4a) with $\xi_* = 1$. We thus find that (3.8) represents the solution of (3.3) satisfying (3.4a) with $\xi_* = 1$ [and (3.4b)]. This is the solution we require. We leave discussion of other values of ξ_* to section 4.

It follows from (3.8) and (3.6b) that

$$\xi(\Xi, \tau) = \Xi + \operatorname{erf}(\Xi/2\tau^{\frac{1}{2}})\,. \tag{3.11}$$

Rearranging (3.8) as

$$\Xi = \pm 2\left[\tau \ln \frac{\vartheta}{(\pi\tau)^{\frac{1}{2}}(1-\vartheta)}\right]^{\frac{1}{2}},$$

we find that (3.11) yields the solution explicit in ξ,

$$|\xi(\vartheta,\tau)| = 2\left[\tau \ln \frac{\vartheta}{(\pi\tau)^{\frac{1}{2}}(1-\vartheta)}\right]^{\frac{1}{2}} + \operatorname{erf}\left[\ln \frac{\vartheta}{(\pi\tau)^{\frac{1}{2}}(1-\vartheta)}\right]^{\frac{1}{2}}. \tag{3.12}$$

The central concentration follows from the result for $\vartheta(0,\tau)$:

$$\vartheta(0,\tau) = [1+(\pi\tau)^{-\frac{1}{2}}]^{-1}. \tag{3.13}$$

Figure 1 gives the time course of the concentration profiles in the reduced form $1-\vartheta(\xi,\tau)$.

4. Nonlinear instantaneous source: discussion

4.1. *Values of* ξ_*

We consider separately the three cases:

(a) $0 \leqq \xi_* < 1$;

(b) $\xi_* = 1$;

(c) $\xi_* > 1$.

(a) No solution exists for $0 \leqq \xi_* < 1$. With $\theta_0 < b$, it is impossible to impose a step-function source with instantaneous concentration greater than b. The instantaneous point source with infinite concentration is the extreme member of this forbidden class of sources.

(b) Our solution for $\xi_* = 1$ represents the distributed instantaneous source at concentration b. This is the most concentrated source possible when $\theta_0 < b$. The solution for this case is much simpler, and therefore potentially more useful, than that for less concentrated distributed sources.

(c) The case $\xi_* > 1$ holds for distributed instantaneous sources with concentration less than b. The relevant exact solution was found in [7]. It is expressible only in implicit form, which somewhat limits its utility. Diffusion from sources with initial concentration less than b may often be estimated readily by a suitable use of the solution for $\xi_* = 1$ (subsection 4.3 below).

4.2. *Physical applicability of solution for* $\xi_* = 1$

Diffusivity function (2.9) is relevant to phenomena where D increases with θ and reaches a very large value at some saturation concentration (i.e. maximum physically possible value of θ). These circumstances hold in their essentials for the phenomena of nonhysteretic water movement in unsaturated nonswelling porous media [1, 3 and, in particular, 13].

It is readily shown, however, that singularities in D stronger than $|\theta - b|^{-n}$ ($n < 1$) are unacceptable physically other than for the instantaneous source. The singularity in (2.9) is stronger than this, but no difficulty or limitation arises in the applications, since b naturally lies outside the physical θ-range. In systems described by (2.1), (2.9), a source at initial concentration b has at least the same level of physical realism as does the classical diffusion source at initial infinite concentration.

For nonhysteretic soil-water movement, a typical value for the saturation moisture content, θ_{sat}, is 0.500; a typical "dry" initial moisture content, θ_0, is 0.200; and the value $D(\theta_{sat})/D(\theta_0) = 3400$ is characteristic. These values require $b = 0.509$. The corresponding value of $\vartheta(0,\tau)$ is then 0.0177, which occurs for $\tau \simeq 0.0001$. It will be seen from Figure 1 that the profile for $\tau = 0.0001$ is almost indistinguishable from a step-function. Our solution would in this case hold for redistribution from a virtual step-function with $\theta = \theta_{sat}$, with a minor correction to the origin of τ.

4.3. *Redistribution from other initial distributions*

The profiles yielded by our solution vary systematically from a step-function at $\tau=0$ to a Gaussian in the limit as $\tau\to\infty$. We thus have the opportunity to match initial distributions of various intermediate shapes. Fitting the initial distribution to the appropriate solution profile characterized by a value of $\vartheta(0,\tau)$ fixes b, so that, with this procedure, the one assignable parameter for matching diffusivity data is a.

4.4. *Case of infinite b*

If $\lim_{b\to\infty} ab^{-2}=\text{constant}=c(0<c<\infty)$, our problem reduces to one in linear diffusion. We find that, in this limit, $x_*\to 0$ and the foregoing solution approaches the classical instantaneous point source solution, as it should.

5. Nonlinear redistribution in the finite region

The nonlinear diffusion equation is mathematically less amenable in the finite region than in the semi-infinite and infinite ones. The only exact unsteady solutions of (2.1) in the finite region known hitherto are somewhat artificial and are valid only for a finite time interval. One class of such solutions can be constructed by truncating the region of the instantaneous source solution $D\propto\theta^n$ $(n>0)$ [8, 9, 10]; a second class can be constructed by truncating appropriate exact solutions in the semi-infinite region which follow from the inverse method of Philip [14]*; and a third class follows similarly from exact solutions in the infinite range [14]**. We present here the first unsteady nonlinear solutions in the finite region in explicit exact form which hold for $0<\tau\leqq\infty$ and which are free of the physically artificial requirement that $D=0$ for at least one point of the relevant θ-range***.

5.1. *General solution*

We consider the redistribution by nonlinear diffusion of an initial distribution of diffusate in the finite region of length l, bounded by surfaces across which no flow is possible. We seek the solution of (2.1), (2.9) subject to the conditions

$$t=0\,,\quad 0\leqq x\leqq l\,,\quad \theta=\theta_0(x)\,; \tag{5.1a}$$

$$t>0\,,\quad x=0 \text{ and } l\,,\quad \partial\theta/\partial x=0\,. \tag{5.1b}$$

The initial distribution $\theta_0(x)$ is essentially arbitrary, though we require that it be a monotonic function of x (and, without further loss of generality, monotonic decreasing). We also require that $\theta_0<b$.

The substitutions

$$\bar{\theta}_0=l^{-1}\int_0^l\theta_0(x)dx\,,\quad \vartheta=\frac{b-\theta}{b-\bar{\theta}_0}\,,\quad \vartheta_0=\frac{b-\theta_0}{b-\bar{\theta}_0}\,,\quad \xi=\frac{\pi x}{l}\,,\quad \tau=\frac{\pi^2 at}{l^2(b-\bar{\theta}_0)^2}\,, \tag{5.2}$$

reduce (2.1), (2.9), (5.1a), (5.1b), to (3.3) subject to the conditions

$$\tau=0\,,\quad 0\leqq\xi\leqq\pi\,,\quad \vartheta=\vartheta_0(\xi)\,; \tag{5.3a}$$

$$\tau>0\,,\quad \xi=0 \text{ and } \pi\,,\quad \partial\vartheta/\partial\xi=0\,. \tag{5.3b}$$

Transformations (3.5), (3.6a) then linearize the problem to (3.7) subject to the conditions

$$\tau=0\,,\quad 0\leqq\Xi\leqq\pi\,,\quad \Theta=\Theta_0(\Xi)\,; \tag{5.4a}$$

$$\tau>0\,,\quad \Xi=0 \text{ and } \pi\,,\quad \partial\Theta/\partial\Xi=0\,. \tag{5.4b}$$

* Table 1 of [14] includes 11 of the simplest functional forms of $D(\theta)$ for which this is true.
** Table 3 of [14] includes 11 of the simplest functional forms of $D(\theta)$ for which this is true.
*** The first and second classes require $D\to 0$ at one end of the θ-range; the third class requires $D\to 0$ at *both* ends of the range.

Here $\Theta_0 = \vartheta_0^{-1}$.

The Fourier cosine expansion of Θ_0 is

$$\Theta_0 = 1 + \sum_{n=1}^{\infty} \alpha_n \cos n\Xi \,, \tag{5.5a}$$

with

$$\alpha_n = \frac{2}{\pi} \int_0^{\pi} \Theta_0(\Xi) \cos n\Xi \, d\Xi \,. \tag{5.5b}$$

The required solution of (3.7), (5.4a), (5.4b) is therefore [12, p. 101]

$$\Theta(\Xi, \tau) = 1 + \sum_{n=1}^{\infty} \alpha_n \, e^{-n^2\tau} \cos n\Xi \,. \tag{5.6}$$

It follows from (3.6b) that

$$\xi(\Xi, \tau) = \Xi + \sum_{n=1}^{\infty} \frac{\alpha_n}{n} e^{-n^2\tau} \sin n\Xi \,. \tag{5.7}$$

Equations (5.6), (5.7) represent the exact solution for nonlinear redistribution in the finite region. In general the solution is implicit and $\xi(\Theta, \tau)$ [whence $x(\theta, t)$] is found through numerical elimination of Ξ between (5.6) and (5.7)*.

We observe, however, that truncations of (5.5a) and (5.6) at $n = N$ are reducible to polynomials of degree N in $\cos \Xi$. Since the roots of polynomials of degree 4 and less may be extracted explicitly, it follows that, in the cases $N = 1, 2, 3, 4$, (5.6) may be rewritten as an explicit expression for $\Xi(\Theta, \tau)$. Substitution in (5.7) then yields $\xi(\Theta, \tau)$ [whence $x(\theta, t)$] explicitly. The cases $N = 3$ and 4 are too complicated to be useful; but the cases $N = 1$ and 2 are of interest. We develop these below.

5.2. *Explicit solution in case* $N = 1$

For $N = 1$, we take

$$\Theta_0 = 1 + \alpha_1 \cos \Xi \,, \tag{5.8}$$

with $0 < \alpha_1 < 1$. The inequality ensures that $\theta < b$. The required solution is then

$$\Theta(\Xi, \tau) = 1 + \alpha_1 \, e^{-\tau} \cos \Xi \,. \tag{5.9}$$

Also, from (3.6b),

$$\xi(\Xi, \tau) = \Xi + \alpha_1 \, e^{-\tau} \sin \Xi \,. \tag{5.10}$$

Now, we have from (5.9) that

$$\Xi = \cos^{-1} [\alpha_1^{-1} e^{\tau} (\Theta - 1)] \,,$$

so that we may write (5.10) in the explicit form

$$\xi(\Theta, \tau) = \cos^{-1} [\alpha_1^{-1} e^{\tau} (\Theta - 1)] + [\alpha_1^2 e^{-2\tau} - (\Theta - 1)^2]^{\frac{1}{2}} \,. \tag{5.11}$$

We observe that we may parametize the $\xi(\Theta)$ relations for all α_1 and for all $\tau \geqq 0$ through the quantity

$$u = \alpha_1 \, e^{-\tau} \,. \tag{5.12}$$

We have then that

$$\xi(\Theta, u) = \cos^{-1} [u^{-1} (\Theta - 1)] + [u^2 - (\Theta - 1)^2]^{\frac{1}{2}} \,. \tag{5.13}$$

* Note that the limiting process of subsection 4.4 here yields $\Theta_0 - 1 \to 0$, and the solution approaches that for linear diffusion, as it should.

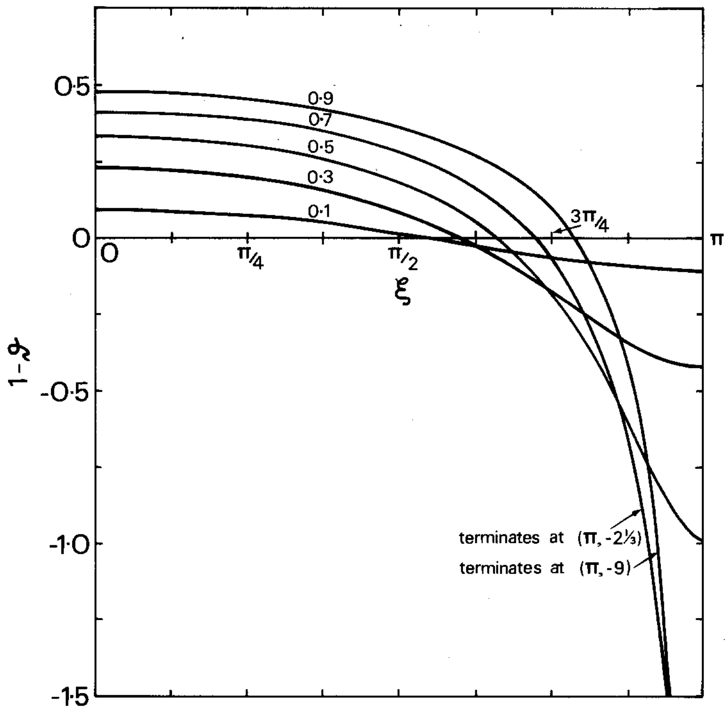

Figure 2. Nonlinear distribution in the finite region: explicit exact solution in case $N=1$. The time course of concentration in the dimensionless form $1-\vartheta(\xi, \tau)$. Note that $1-\vartheta=(\theta-\bar{\theta}_0)/(b-\bar{\theta}_0)$. ξ is the reduced space coordinate. $u=\alpha_1 e^{-\tau}$, where τ is the reduced time, and the initial distribution, in reduced form, is given by $\xi=\cos^{-1}[\alpha_1(\vartheta^{-1}-1)]+[\alpha_1^2-(\vartheta^{-1}-1)^2]^{\frac{1}{2}}$. Numerals on the curves denote values of u. The time course of redistribution for given α_1 is represented by the sequence of curves $u \leqq \alpha_1$.

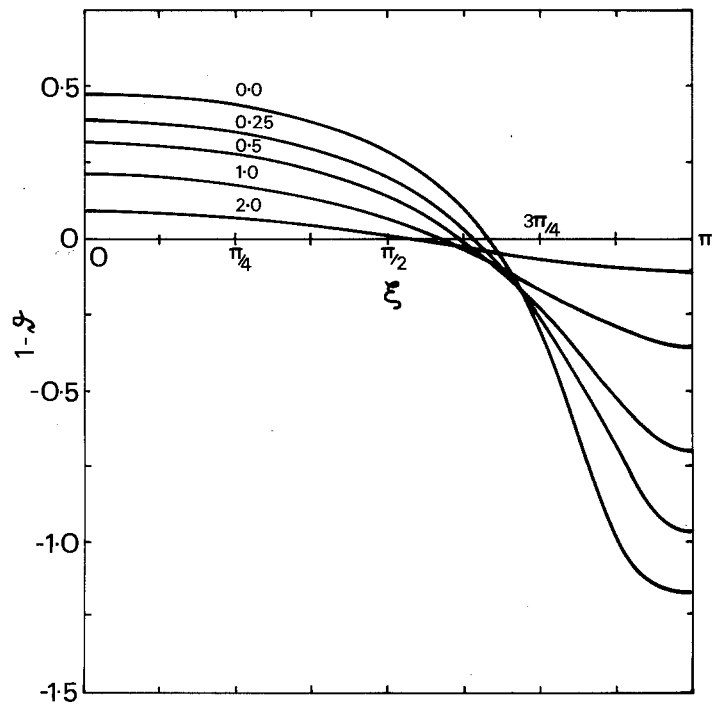

Figure 3. Nonlinear distribution in the finite region: explicit exact solution in case $N=2$. The time course of concentration in the dimensionless form $1-\vartheta(\xi, \tau)$. Note that $1-\vartheta=(\theta-\bar{\theta}_0)/(b-\bar{\theta}_0)$. ξ is the reduced space coordinate. Numerals on the curves denote values of the reduced time τ. Solution for $\alpha_1=0.72$, $\alpha_2=0.18$.

For given u, ξ goes from 0 to π as Θ goes from $1+u$ to $1-u$. For any given α_1, the solution $x(\theta, t)$, of course, follows at once from (5.11) or (5.13). Figure 2 graphs the solution in the form $(\theta-\theta_0)/(b-\bar{\theta}_0)\,[=1-\vartheta=1-\Theta^{-1}]$ against ξ for various u. The time course of the redistribution process corresponding to a particular α_1-value is represented by the sequence of curves with $u \leqq \alpha_1$, the curve $u=\alpha_1$ representing (in reduced form) the appropriate $\theta_0(x)$.

5.3. *Explicit solution in case $N=2$*

For $N=2$ we take

$$\Theta_0 = 1+\alpha_1 \cos \Xi+\alpha_2 \cos 2\Xi\,. \tag{5.14}$$

The inequality $0<\alpha_1-\alpha_2<1$ ensures that $\theta<b$; and the further inequality $|\alpha_2|\leqq\frac{1}{4}\alpha_1$ ensures that $\theta_0(x)$ is monotonic decreasing. The required solution is then

$$\Theta(\Xi, \tau) = 1+\alpha_1 e^{-\tau} \cos \Xi+\alpha_2 e^{-4\tau} \cos 2\Xi\,, \tag{5.15}$$

whence, from (3.6b),

$$\xi(\Xi, \tau) = \Xi+\alpha_1 e^{-\tau} \sin \Xi+\tfrac{1}{2}\alpha_2 e^{-4\tau} \sin 2\Xi\,. \tag{5.16}$$

It follows from (5.15) that

$$\begin{aligned} \Xi &= \cos^{-1}\left[\tfrac{1}{4}\alpha_2^{-1} e^{3\tau}\{[\alpha_1^2+8\alpha_2 e^{-2\tau}(\Theta-1)+8\alpha_2^2 e^{-6\tau}]^{\frac{1}{2}}-\alpha_1\}\right] \\ &= \cos^{-1}[g(\Theta, \tau)]\,. \end{aligned} \tag{5.17}$$

The explicit form of (5.15) is therefore

$$\xi(\Theta, \tau) = \cos^{-1} g+(\alpha_1 e^{-\tau}+\alpha_2 g e^{-4\tau})(1-g^2)^{\frac{1}{2}}\,. \tag{5.18}$$

Note that ξ goes from 0 to π as Θ goes from $1+\alpha_1 e^{-\tau}+\alpha_2 e^{-4\tau}$ to $1-\alpha_1 e^{-\tau}+\alpha_2^{-4\tau}$.

Calculations for $N=2$ are more complicated than those for $N=1$, but two-parameter representation of Θ_0 naturally provides greater flexibility in fitting initial distributions. A useful class of shapes of Θ_0 is generated by taking $\alpha_2=\frac{1}{4}\alpha_1$. In this case there are some simplifications to the algebra.

Figure 3 shows the solution for the case $\alpha_1=0.72$, $\alpha_2=0.18$, in the form $(\theta-\bar{\theta}_0)/(b-\bar{\theta}_0)$ against ξ for various τ. The curve for $\tau=0$ should be compared with that for $u=0.9$ in Figure 2. Both curves have $(\theta_0(0)-\bar{\theta}_0)/(b-\bar{\theta}_0)=0.474$. The differences between them are indicative of the range of $\theta_0(x)$ shapes which may be represented.

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